Comparison of Four Alternative Fixed Point Algorithms for Applied General Equilibrium Modeling

Hidetaka Kawano

A new globally convergent price adjustment process has recently been proposed by Joosten and Talman. The main strengths of this approach are its globally convergent characteristics and the possibility of its application to a large-scale applied general equilibrium model. In order to see the empirical applicability of the Joosten-Talman approach, for this paper I conducted a numerical simulation of the simple illustrative model and compared its performance with three alternative approaches. The best approach was the modified Kimbell-Harrison approach which achieved a high computational precision: only 20 iterations were needed to obtain the converged values. The Joosten-Talman approach also achieved a high precision, requiring just 30 iterations. For empirical models, both approaches appear promising.

JEF classification: C63; C68; D58
Keywords: Applied general equilibrium; Algorithm; Globally convergent price adjustment

1 Introduction

At the outset of its theoretical formulation in the 1950s, the major criticism of the general equilibrium framework was its intractability in dealing with multi-dimensional empirical issues. The framework was said to lack efficiently operational algorithms and sufficient computational power. This criticism was significantly weakened with the first applications of the Johansen approach in the early 1960s (Johansen [2, 1960] and Dixon, Parmenter, and Powell [1, 1992]), the Scarf approach in late 1960s and early 1970s (Scarf [6, 1967] and Scarf [7, 1973]). Subsequent refinements in operational algorithms have been made, while at the same time computational power has been considerably increased. Thus, the early criticism of general equilibrium analysis no longer holds. The number of such empirical models, called "applied general equilibrium models"(or sometimes called "computable general equilibrium model") has mushroomed in various fields of research in the last 20 years.

A new globally convergent price adjustment process has recently been proposed by Joosten and Talman [3, 1998]. The main strengths of this approach are its globally convergent characteristics and the possibility of its application to a large-scale applied general equilibrium model. In order to see the empirical applicability of the Joosten-Talman approach (Joosten and Talman [3, 1998]), for this paper I conducted a numerical simulation of the simple illustrative model and compared its performance with three alternative approaches: the Scarf approach (Scarf [6, 1967], Scarf [7, 1973], and Shoven and Whalley [8, 1992]), the Modified Tanaka-Kawano approach (Tanaka and Kawano [9, 1996]), and the Modified Kimbell-Harrison approach (Kimbell-Harrison [5, 1986] and Kawano [4, 2003]). The challenge for empirical application is to develop an easily implementable algorithm which is relatively

fast and efficient in computing in high precision. The computational example for this paper was drawn from Washida [10, 2004: 256-7] who provided the ideal structure for testing the numerical reliability of each approach. Washida [10, 2004:244-9 and 256-67] also provided his program and an excellent exposition of the Scarf algorithm.

The simulation exercise revealed the best approach was the modified Kimbell-Harrison approach, which achieved a high computational precision, needing only 20 iterations to obtain the converged values. The Joosten-Talman approach also achieved a high precision, requiring just 30 iterations. The modified Tanaka-Kawano approach achieved a high precision, but only after 178 iterations. The Scarf approach performed poorly and did not achieve a high precision even with 1 million iterations; its computational time was 1.978 seconds, which is comparatively long. The other approaches took less than 0.001 second to converge. These experiments were conducted using Intel's 333MHz Pentium II processor and were programmed in C-language. The verified reliability of the simulation results in double precision (1.0e-15).

In section 2, the general structure of the model is specified. In section 3, the four alternative approaches are described. In section 4, the simulation results are summarized. The conclusion follows in section 5. The complete computer output is presented in the Appendix.

2 Model Structure

The model is a simple competitive pure exchange economy with 3 commodities indexed $j \in I^3 \equiv \{1,2,3\}$ and two consumers indexed $i \in I^2 \equiv \{1,2\}$. The set of all possible price vectors is $\Re^3_+ \setminus \{0^3\}$, where $\Re^3_+ \equiv \{x \in \Re^3 \mid x_j \geq 0\ \forall_j \in I^3\}$. Each consumer $i \in I^2$ is characterized by his consumption set X^i, initial endowments ω^i, and preference relation $\preceq^i$ with the following 3 assumptions:

Assumption 1: The consumption set X^i is a compact, convex subset of $\Re^3_+$, containing the set $\left\{x \in \Re^3 \mid 0 \leq x_j \leq \Sigma^2_{i=1}\ \omega^i_j,\ \forall_j \in I^3\right\}$, where $\omega^i = (\omega^i_1, \omega^i_2, \omega^i_3)$ is the vector of initial endowments of the 3 commodities $j \in I^3$ of consumer $i \in I^2$.

Assumption 2: $\omega^i_j > 0,\ \forall\ i \in I^2,\ j \in I^3$.

Assumption 3: The preference relation $\preceq^i$ is continuous, monotonic, and strictly convex.

Let $B^i(p) = \{x \in X^i \mid p^T x \leq p^T \omega^i\}$ denote the budget set of consumer $i \in I^2$ given price vector $p \in \Re^3_+ \setminus \{0^3\}$. By assumption, each consumer $i \in I^2$ maximizes his utility function over his budget set. Under Assumptions 1 through 3, the solution of each consumer's maximization problem is unique and also satisfies the budget constraint with equality: $p^T d^i(p) = p^T \omega^i$, for each consumer $i \in I^2$, for every $p \in \Re^3_+ \setminus \{0^3\}$.

The derived demand function of each consumer $i \in I^2$ denoted by $d^i(p)$: $\Re^3_+ \setminus \{0^3\} \rightarrow \Re^3$ is continuous. The excess demand function $z^i(p)$ of consumer $i \in I^2$ is defined as: $z^i(p) = d^i(p) - \omega^i$, for every $p \in \Re^{n+1}_+ \setminus \{0^{n+1}\}$. Then, $p^T z^i(p) = 0$ for every $p \in \Re^3_+ \setminus \{0^3\}$. The aggregate excess demand function z: $\Re^3_+ \setminus \{0^3\} \rightarrow \Re^3$ defined as: $z(p) \equiv \Sigma_{i \in I^2}\ z^i(p)$, for every $p \in \Re^3_+ \setminus \{0^3\}$ is a continuous function with the following properties:

Property 1: $p^T z(p) = 0,\ \forall\ p \in \Re^3_+ \setminus \{0^3\}$ (Walras' law);
Property 2: $z(p) > 0$, whenever $p_j = 0$ (desirability);
Property 3: $z(\mu \cdot p) = z(p)$, $\forall$ *scalars* $\mu > 0$, $\forall\ p \in \Re^3_+ \setminus \{0^3\}$ (homogeneity of degree zero in prices).

Note that homogeneity of degree zero in prices allows normalization of the price space

to the n-dimensional unit simplex. Therefore the analysis of aggregate excess demand functions is restricted to the normalized price space formed by the unit simplex. The n-dimensional unit simplex S^n is defined by:

$$S^n \equiv \left\{ p_j \in \Re_+^{n+1} \mid \sum_{j+1}^{n+1} p_j = 1 \right\}$$

In this example, $n = 2$. An equilibrium price vector is a price vector denoted by $p^* \in S^n$, where aggregate excess demand function $z(p^*) = 0^{n+1}$ holds for each commodity. It is well-known that such an equilibrium price vector p^* always exists on the n-dimensional unit simplex satisfying $z(p^*) = 0^{n+1}$.

In this model structure, the economy $\varepsilon = ((X^i, \preceq^i, \omega^i)_{i \in I^i})$ is such that the compact consumption set $X^{i \in I^i} \subset \Re_+^3$. Each consumer's preference relations $\preceq^1$ and $\preceq^2$ are represented by utility functions $u^1 : X^1 \to \Re$ and $u^2 : X^2 \to \Re$, respectively, defined by

$$u^1(x^1) = (x_1^1)^{0.3} (x_2^1)^{0.3} (x_3^1)^{0.4}, \ \forall x^1 \in \Re_+^2,$$
$$u^2(x^2) = (x_1^2)^{0.5} (x_2^2)^{0.2} (x_3^2)^{0.3}, \ \forall x^2 \in \Re_+^2.$$

The vectors of initial endowments of the 3 commodities $j \in I^3$ of consumer $i \in I^2$ are given as:

$$\omega^1 = (\omega_1^1, \omega_2^1, \omega_3^1)^T = (40, 20, 30)^T,$$
$$\omega^2 = (\omega_1^2, \omega_2^2, \omega_3^2)^T = (20, 30, 40)^T,$$
$$\sum_{i=1}^{2} \omega^i = (60, 50, 70)^T.$$

Each consumer's income denoted by $Y^i \ \forall \ i \in I^2$ is given as follows:

$$Y^i = p^T \omega^i.$$

This example is taken from Washida [10, 2004:256-7]. As a result of each consumer's utility maximization, the corresponding demand functions for consumer $i \in I^2$ are denoted by $d^i(p)$, $\forall \ i \in I^2$ as:

$$d^1(p) = \left(\frac{0.3Y^1}{p_1}, \frac{0.3Y^1}{p_2}, \frac{0.4Y^1}{p_3} \right)^T, \forall \, p \in \Re_+^3 \setminus \{0^3\},$$

$$d^2(p) = \left(\frac{0.5Y^2}{p_1}, \frac{0.2Y^2}{p_2}, \frac{0.3Y^2}{p_3} \right)^T, \forall \, p \in \Re_+^3 \setminus \{0^3\},$$

It is also easily verified that for every consumer $i \in I^2$, $\forall \ i \in I^2$ is a continuous function. The Walrasian equilibrium price systems are given by the solutions $p^* \in \Re_+^3 \setminus \{0^3\}$ of the system of equations of the aggregate demand function $z(p)$: $\Re_+^3 \setminus \{0^3\} \to \Re^3$ as:

$$z(p) \equiv \sum_{i \in I^2} d^i(p) - \sum_{i \in I^2} \omega^i = 0.$$

A Walrasian equilibrium price system $p^* = (p_1^*, p_2^*, p_3^*)^T$ satisfying the condition $\Sigma_{j=1}^3 p_j = 1$ is numerically computed (by the four alternative approaches chosen in this paper) if and only if $p^* \in \Re_+^3 \setminus \{0^3\}$. The unique Walrasian equilibrium price vector p^* is given by:

$$p^* = (0.396791, \ 0.30187, \ 0.301333)^T.$$

The computational results for four alternative approaches are presented in the Appendix. All assumptions made are satisfied by this economy ε.

3 Descriptions of the Four Alternative Approaches

In this section, the four algorithms applied to compute the benchmark model in Section 2 can be briefly described as follows:

1. Scarf Approach: This approach was first introduced by Scarf [6, 1967], and the exposition follows Shoven and Whalley [8, 1992:40-1]. To find such a fixed point, the unit simplex is divided into a finite number of smaller simplices, each defined by N vertices that are each associated with a label. These labels are chosen from the set of integers that

defined the dimensionality of the general equilibrium problem (if N is the dimensionality of the problem). The labels are chosen in such a way that if a simplex can be found whose vertices have a complete set of labels (from 1 to N) associated with them, then this implies that a close approximation to a general equilibrium must have been found. The operational program used in this paper was developed by Washida [10, 2004:256-7] and modified for the implementation.

2. Joosten-Talman Approach: This approach was developed by Joosten and Talman [3, 1998:15-26]. The purpose of the paper was to introduce a new globally convergent price adjustment process. Under the price adjustment process, only the prices of commodities having the highest excess demand are increased from their initial levels. At the same time, only the prices of commodities having the lowest excess demand are decreased from their initial levels. This process was easily programmed for operational use.

3. Modified Kimbell-Harrison Approach: This approach was developed by Kimbell and Harrison [5, 1986:197-212] and extended by Kawano [4, 2003:2-27]. The crucial step for revising prices over iterations is to increase prices for all the excess demands and to lower all the excess supplies simultaneously. This Walrasian tatonnament process was simply formulated in the program as : $p_{n+1} = p_n (D/S)$, where $D :=$ excess demand and $S :=$ excess supply.

4. Modified Tanaka-Kawano Approach: This approach was developed for solving a general equilibrium model by Tanaka and Kawano [9, 1996] and extended to solve the multi-dimensional case. The main feature is: if the sign of the same excess demand function changes, reduce the currently assigned incremental step length by half, if the absolute value of the excess demand function is increased, then reverse the direction of the incremental step length; if otherwise, reassign the same step length and continues the iteration.

4 Simulation Results

Among the four alternative approaches starting with the same initial values, the best approach was the modified Kimbell-Harrison approach which achieved a high computational precision, needing only 20 iterations to obtain the converged values (see Program wsd-1.c in the Appendix). The Joosten-Talman approach also achieved a high precision, requiring just 30 iterations (see Program wsd-3.c in the Appendix). The modified Tanaka-Kawano approach achieved a high precision, but only after 178 iterations (see Program wsd-7.c in the Appendix). The Scarf approach performed poorly and did not achieve a high precision even with 1 million iterations; its computational time was 1.978 seconds which is comparatively long (see Program scarf 3.c in the Appendix). The other approaches took less than 0.001 second to converge. Even though the Scarf approach did poorly in this simulation, its discretized grid method could be used initially to search for approximately converged values over the entire n-dimensional unit simplex. Then, these approximately converged values could be fed into the other higher precision algorithms. These experiments were conducted using Intel's 333MHz Pentium II processor and were programmed in C-language. The verified reliability of the simulation results in double precision (1.0e-15).

5 Conclusion

In order to investigate the empirical applicability of the Joosten-Talman approach (Joosten and Talman [3, 1998]), for this paper I conducted a numerical simulation of the simple illustrative model and compared its performance with three alternative approaches. The simulation exercise revealed the best approach was the modified Kimbell-Harrison approach which achieved a high computational precision; only 20 iterations were needed to obtain the converged values. The Joosten-Talman approach also achieved a high precision, requiring just 30 iterations. They are both considered the most promising approaches for empirical models.

(Aomori Public College)

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APPENDIX

A Computational Results

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======================The converged values================================

 PROGRAM: wsd-1.c
 ALGORITHM: Modified Kimbell-Harrison

 /// 1. Computational time & Iterations ///

 1-1)  Initial values: p[0]=0.999998
                       p[1]=0.000001

 1-2)  Iteration for general equilibrium loop: No.= 20
 1-3)  Computational time:    0.000 seconds have passed.


 /// 2. Commodity prices p[i] ///

 2-1)  p[0]= 0.396790862115855
 2-2)  p[1]= 0.301876529779712
 2-3)  p[2]= 0.301332608104433


 /// 3. Excess demands for commodity markets rho[i] ///

 3-1)  rho[0]=-0.000000000000007
 3-2)  rho[1]= 0.000000000000000
 3-3)  rho[2]= 0.000000000000014

==========================The end of the output flie=======================


======================The converged values================================

 PROGRAM: wsd-7.c
 ALGORITHM: Modified Tanaka-Kawano

 /// 1. Computational time & Iterations ///

 1-1)  Initial values: p[0]=0.999998
                       p[1]=0.000001

 1-2)  Iteration for general equilibrium loop: No.= 178
 1-3)  Computational time:    0.000 seconds have passed.


 /// 2. Commodity prices p[i] ///

 2-1)  p[0]= 0.396790862115855
 2-2)  p[1]= 0.301876529779712
 2-3)  p[2]= 0.301332608104433


 /// 3. Excess demands for commodity markets rho[i] ///

 3-1)  rho[0]=-0.000000000000014
 3-2)  rho[1]= 0.000000000000000
 3-3)  rho[2]= 0.000000000000028

 ==========================The end of the output flie=======================
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```
=================The converged values=======================

 PROGRAM: scarf3.c
 ALGORITM: Scarf

 /// 1. Computational time & Iterations ///

 1-1)  Initial values: p[0]=0.999998
                       p[1]=0.000001

 1-2)  Iteration for general equilibrium loop: No.= 1206418
 1-3)  Computational time:    1.978 seconds have passed.


 /// 2. Commodity prices Prices[i] ///

 2-1)  p[0]= 0.396790000000000
 2-2)  p[1]= 0.301877000000000
 2-3)  p[2]= 0.301333000000000


 /// 3. Excess demands for commodity markets rho_L[k],rho_K[k] ///

 3-1)  rho[0]= 0.000136092139421
 3-2)  rho[1]=-0.000082815186317
 3-3)  rho[2]=-0.000096239044510

======================The end of the output flie==================


======================The converged values=================================

 PROGRAM: wsd-3.c
 ALGORITHM: Joosten-Talman

 /// 1. Computational time & Iterations ///

 1-1)  Initial values: p[0]=0.999998
                       p[1]=0.000001

 1-2)  Iteration for general equilibrium loop: No.= 30
 1-3)  Computational time:    0.000 seconds have passed.


 /// 2. Commodity prices p[i] ///

 2-1)  p[0]= 0.396790862115855
 2-2)  p[1]= 0.301876529779712
 2-3)  p[2]= 0.301332608104433


 /// 3. Excess demands for commodity markets rho[i] ///

 3-1)  rho[0]=-0.000000000000007
 3-2)  rho[1]= 0.000000000000000
 3-3)  rho[2]= 0.000000000000014

=========================The end of the output flie=======================
```